MATH 53, Practice for Midterm 1

You should allocate 90 minutes to do the following 9 problems (starting on the back of this page). The difficulty and spread of topics are *not* indicative of the actual midterm. Most of the problems are exercises from Stewart (and I expect the actual midterm to be like that as well).

Make sure to show your reasoning, as an answer with no explanation will receive no credit on the actual exam. It is also a good habit to box your final answers.

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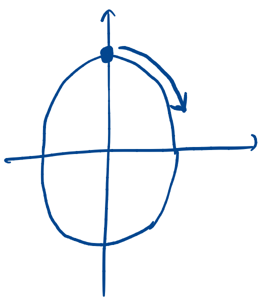
Problem 1 (adapted from §10.2.31). Let $a, b > 0$. The equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

defines an ellipse.

a) Parametrize this ellipse so that it starts and ends at $(0, b)$, and traces the curve out once clockwise.

Want to use sin and cos. Note that x starts at 0 and increases, while y starts at b and decreases. So let's use



$$x = a \sin t \qquad 0 \le t < 2\pi$$
$$y = b \cos t$$

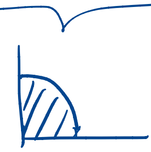
b) Compute the area enclosed by the ellipse.

$$\text{Area} \overset{(*)}{=} \int_0^{2\pi} \underbrace{(b \cos t)}_{y} \underbrace{\frac{d}{dt}(a \sin t)\, dt}_{dx} = \int_0^{2\pi} ab \cos^2 t \, dt$$

$$= ab \int_0^{2\pi} \frac{1 + \cos 2t}{2} \, dt = ab \left(\frac{t + \frac{1}{2}\sin 2t}{2} \right) \Bigg|_{t=0}^{2\pi} = \boxed{ab\pi}$$

I explained in discussion why $(*)$ works to compute the area enclosed by a loop.

Alternatively one could compute $4 \cdot \underbrace{\int_0^{\pi/2} (b \cos t) \frac{d}{dt}(a \sin t)\, dt}$ by symmetry.



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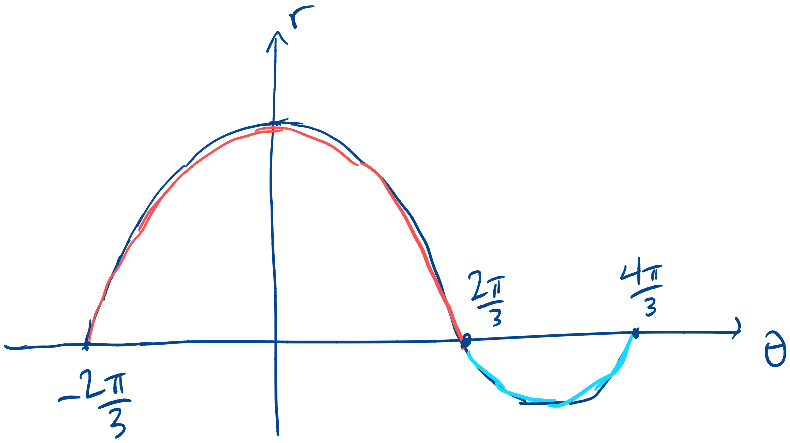
Problem 2 (§10.4.35). Find the area inside the larger loop and outside the smaller loop of the limaçon $r = \frac{1}{2} + \cos\theta$.

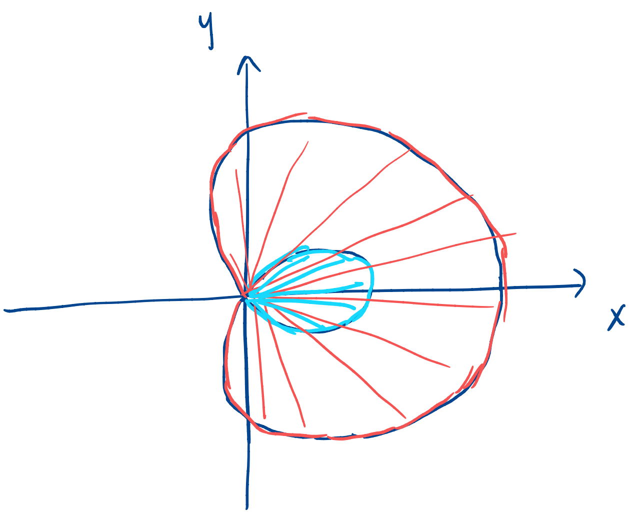


Find angles where it goes through origin, i.e. $r = 0$.

$$\frac{1}{2} + \cos\theta = 0$$

$$\cos\theta = -\frac{1}{2}.$$

$$(*) \quad \int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{2}\left(\frac{1}{2}+\cos\theta\right)^2 d\theta - \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}\left(\frac{1}{2}+\cos\theta\right)^2 d\theta$$



Evaluate $\displaystyle\int \frac{1}{2}\left(\frac{1}{2}+\cos\theta\right)^2 d\theta = \frac{1}{2}\int\left(\frac{1}{4}+\cos\theta+\cos^2\theta\right)d\theta$

$$= \frac{1}{2}\left(\frac{\theta}{4} + \sin\theta + \int \frac{1+\cos 2\theta}{2}\, d\theta\right)$$

$$= \frac{1}{2}\left(\frac{\theta}{4} + \sin\theta + \frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right)$$

$$= \frac{3}{8}\theta + \frac{1}{2}\sin\theta + \frac{1}{8}\sin 2\theta \quad \leftarrow \text{call this } g(\theta)$$

$(*)$ is equal to $g\left(\frac{2\pi}{3}\right) - g\left(-\frac{2\pi}{3}\right) - g\left(\frac{4\pi}{3}\right) + g\left(\frac{2\pi}{3}\right).$

$$= 3g\left(\frac{2\pi}{3}\right) - g\left(\frac{4\pi}{3}\right)$$

$$= 3\left(\frac{3\sqrt{3}}{16} + \frac{\pi}{4}\right) - \left(\frac{\pi}{2} - \frac{3\sqrt{3}}{16}\right)$$

$$= \boxed{\frac{1}{4}\left(3\sqrt{3}+\pi\right)}$$

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Problem 3. Consider the line L_1 given by

$$\frac{x-1}{3} = \frac{y-4}{2} = \frac{z+3}{-4} = t \rightsquigarrow \langle 3t+1, 2t+4, -4t-3 \rangle$$

and the line L_2 given by

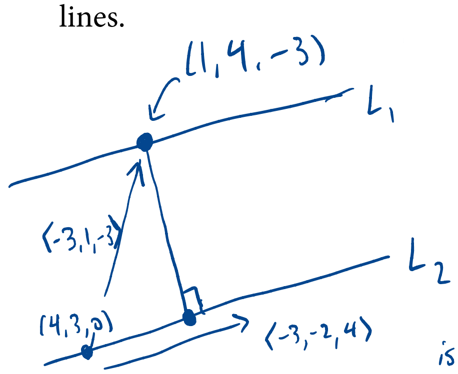
$$\mathbf{r}(t) = \langle 4-3t, 3-2t, 4t \rangle.$$

a) Are L_1 and L_2 parallel, intersecting, or skew?

dir. vector for L_1: $\langle 3, 2, -4 \rangle$. for L_2: $\langle -3, -2, 4 \rangle$.

These are multiples of each other, so the lines are parallel.

b) If they intersect, find the point at which they intersect. Otherwise, determine the distance between the two lines.



Pick any pt on L_1, such as $(1,4,-3)$.

The distance from L_1 to L_2 is just the distance from $(1,4,-3)$ to L_2. (This only is valid when the lines are parallel!)

There are a lot of ways of doing the problem from here. Here is one:

$$\frac{|\langle -3,-2,4 \rangle \times \langle -3,1,-3 \rangle|}{|\langle -3,-2,4 \rangle|} = \frac{\sqrt{2^2+21^2+9^2}}{\sqrt{3^2+2^2+4^2}} = \sqrt{\frac{526}{29}}$$

(correction to original answer)

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & -2 & 4 \\ -3 & 1 & -3 \end{vmatrix} = \langle 2, -21, -9 \rangle.$$

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Problem 4. Let C be the curve of intersection of the surfaces $z = \sqrt{x^2 + y^2}$ and $z = 1 - y$.

a) Parametrize the curve C.

$$\begin{cases} z^2 = x^2 + y^2 \\ z = 1 - y \end{cases}$$

$$\begin{cases} 1 - 2y + \cancel{y^2} = x^2 + \cancel{y^2} \\ z = 1 - y \end{cases}$$

$$\boxed{\begin{aligned} x &= t \\ y &= \frac{1-t^2}{2} \\ z &= \frac{1+t^2}{2} \end{aligned}}$$

or $\vec{r}(t) = \langle t, \frac{1-t^2}{2}, \frac{1+t^2}{2} \rangle$

So x is a good choice as parameter, b/c y and z are easily expressed in terms of x.

b) Compute an equation for the tangent line to C at the point $(1, 0, 1)$.

This corresponds to $t = 1$ with our parametrization.

$\vec{r}\,'(t) = \langle 1, -t, t \rangle$, $\vec{r}\,'(1) = \langle 1, -1, 1 \rangle$

so tangent line is:

$$\boxed{L(t) = \langle 1, 0, 1 \rangle + t \langle 1, -1, 1 \rangle}$$

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Problem 5.

a) Find a function $f(x,y)$ such that $f_x(x,y) = ye^{xy} + \sin y$ and $f_y(x,y) = xe^{xy} + x\cos y$.

Integrate $f_x(x,y) = ye^{xy} + \sin y$ with respect to x:

$$f(x,y) = e^{xy} + x\sin y + \underbrace{g(y)}$$

"constant" w.r.t. x means depends only on y.

Differentiate this w.r.t. y:

$$f_y(x,y) = xe^{xy} + x\cos y + g'(y).$$

So $g'(y) = 0$, meaning

$g(y) = C$, any constant.

Hence $\boxed{f(x,y) = e^{xy} + x\sin y + 42}$ suffices.

(corrected from original sol)

b) (§14.3.97) On the other hand, there is no $g(x,y)$ such that $g_x(x,y) = x + 4y$ and $g_y(x,y) = 3x - y$. Explain why, *without* using integration.

If such a g existed, then

$$g_{xy}(x,y) = \frac{\partial}{\partial y}(x + 4y) = 4$$

while

$$g_{yx}(x,y) = \frac{\partial}{\partial x}(3x - y) = 3$$

Then $g_{xy} \neq g_{yx}$ contradicting Clairaut's thm.

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Problem 6. Let $f(x, y) = \sqrt{xy}$, and P be the point $(2, 8)$.

a) At the point P, give a unit vector pointing in the direction in which f *decreases* the most rapidly.

This unit vector should point in the direction of $-\nabla f(2,8)$.

$$\nabla f(x,y) = \left\langle \frac{1}{2}(xy)^{-\frac{1}{2}} y,\ \frac{1}{2}(xy)^{-\frac{1}{2}} x \right\rangle.$$

$$\nabla f(2,8) = \left\langle \frac{1}{2}\cdot\frac{1}{4}\cdot 8,\ \frac{1}{2}\cdot\frac{1}{4}\cdot 2 \right\rangle = \left\langle 1, \frac{1}{4} \right\rangle$$

So our unit vector is $\dfrac{-\langle 1, \frac{1}{4}\rangle}{\sqrt{1+\frac{1}{16}}} = \boxed{\left\langle -\frac{4}{\sqrt{17}},\ -\frac{1}{\sqrt{17}} \right\rangle}$

b) (§14.6.19) Compute the directional derivative of f at P in the direction of the point $Q = (5, 4)$.

(correction to original sol.)

$\overrightarrow{PQ} = \langle 3, -4 \rangle$ unit vec $\vec{u}$ in this direction: $\left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle$.

$$D_{\vec{u}} f(P) = \nabla f(2,8)\cdot\vec{u} = \left\langle 1, \frac{1}{4}\right\rangle \cdot \left\langle \frac{3}{5}, -\frac{4}{5}\right\rangle = \frac{3}{5} - \frac{1}{5} = \boxed{\frac{2}{5}}$$

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Problem 7. The point $(2, 3, 6)$ lies on the surface S given by $z = xy$. Let H be the tangent plane to S at that point. Show that the intersection of H with S is a pair of lines, and give their equations.

$f(x,y) = xy.$ $\quad \nabla f = \langle y, x \rangle$

$\nabla f(2,3) = \langle 3, 2 \rangle.$

eqn. of H: $\quad z = 6 + 3(x-2) + 2(y-3) = 3x + 2y - 6$

So the intersection is the system $\begin{cases} z = xy \\ z = 3x + 2y - 6. \end{cases}$

$$\begin{cases} z = xy \\ xy - 3x - 2y + 6 = 0 \end{cases}$$

$$\begin{cases} z = xy \\ (x-2)(y-3) = 0 \end{cases} \Longrightarrow \boxed{\begin{array}{l} x = 2 \\ \text{and } z = 2y \end{array}} \text{ or } \boxed{\begin{array}{l} y = 3 \\ \text{and } z = 3x \end{array}}$$

"Symmetric equations" of the two lines.

converted to vector form:

$L_1(t) = \langle 2, t/2, t \rangle, \quad L_2(t) = \langle t/3, 3, t \rangle$

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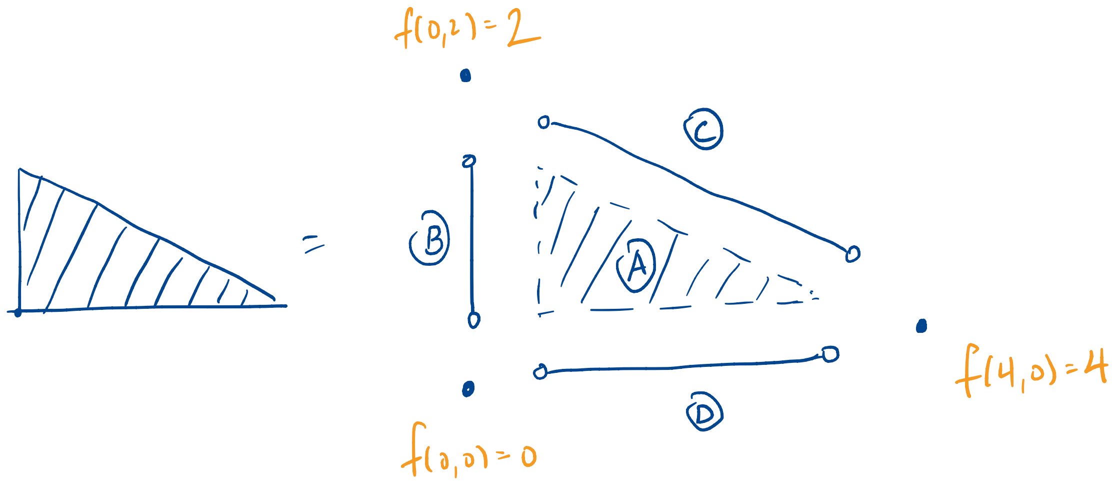
Problem 8 (§14.7.32). Find the absolute maximum and minimum values of

$$f(x, y) = x + y - xy$$

on the closed triangular region with vertices $(0, 0)$, $(0, 2)$, and $(4, 0)$.



A: $\nabla f(x,y) = \vec{0}$

$\langle 1-y, 1-x \rangle = \vec{0}$

$x=1, y=1$ (yes, this is in region A)

$f(1,1) = 1$

C: Restrict to $\langle x,y \rangle = \langle 0,2 \rangle + t\langle 4,-2 \rangle$, $0 < t < 1$

i.e. $x = 4t$
$y = 2 - 2t$

$\frac{d}{dt}\left(f(4t, 2-2t)\right) = 0$

$\frac{d}{dt}\left(4t + 2 - 2t - 8t + 8t^2\right) = 0$

$-6 + 16t = 0$

$t = \frac{3}{8}$ (yes, $0 < \frac{3}{8} < 1$)

$f\left(\frac{3}{2}, \frac{5}{4}\right) = 7/8$

B: Restrict to $x = 0$, $0 < y < 2$

$\frac{d}{dy}\left(f(0,y)\right) = 0$

$\frac{d}{dy}(y) = 0$ no solutions.

$1 = 0$

D: Restrict to $0 < x < 4$, $y = 0$.

$\frac{d}{dx}\left(f(x,0)\right) = 0$

$1 = 0$ no solutions

Compare values: $\boxed{\begin{array}{l}\text{min is } 0 \text{ attained @ } (0,0) \\ \text{max is } 4 \text{ attained @ } (4,0)\end{array}}$

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Problem 9. Let $z = f(x, y)$, where f is a function satisfying the identity

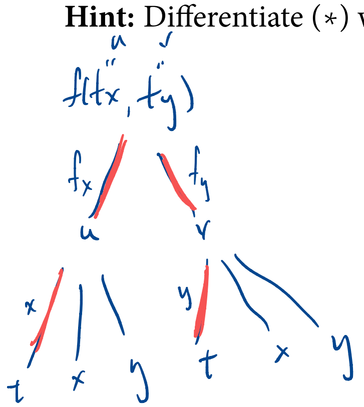
$$f(tx, ty) = t^3 f(x, y) \tag{$*$}$$

for all t.

a) (Adapted from §14.5.55) Assume for this part that f is a very "nice" differentiable function. Show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z.$$

Hint: Differentiate $(*)$ with respect to t.



So differentiating $(*)$ w.r.t. t gives

$$x f_x(tx, ty) + y f_y(tx, ty) = 3t^2 f(x,y)$$

Now plug in $t=1$ to get the desired result:

$$x f_x(x,y) + y f_y(x,y) = 3f(x,y).$$

b) (Hard; do not attempt unless you are done with the rest of the exam.) Suppose that, aside from $(*)$, we only know that the domain of f is all of $\mathbb{R}^2$, and that f is continuous at all points other than $(0, 0)$. Prove that f is also continuous at $(0, 0)$. **Hint:** try switching to polar to evaluate the limit.

Note that $f(0,0) = f(t\cdot 0, t\cdot 0) = t^3 f(0,0)$ for all t, so $f(0,0) = 0$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{\substack{r\to 0^+ \\ \theta \text{ any}}} f(r\cos\theta, r\sin\theta) = \lim_{\substack{r\to 0^+ \\ \theta \text{ any}}} r^3 f(\cos\theta, \sin\theta).$$

The points $(\cos\theta, \sin\theta)$ are on the unit circle, which is closed and bounded. Since f is continuous on the unit circle, EVT means it attains a max M and a min m.

$$m r^3 \le r^3 f(\cos\theta, \sin\theta) \le M r^3 \qquad (r > 0)$$

Then, since $\lim\limits_{\substack{r\to 0^+ \\ \theta \text{ any}}} m r^3 = 0 = \lim\limits_{\substack{r\to 0^+ \\ \theta \text{ any}}} M r^3$, the Squeeze Thm. tells us

$\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$. Hence the limit is equal to $f(0,0)$ and f is continuous @ $(0,0)$.